Calc I: Worksheet 3

Name: ____________________________

1. (a) Find the slope of the tangent to the curve at $y = \frac{1}{\sqrt{x}}$ at the point where $x = a$.

   (b) Find equations of the tangent lines at the points $(1, 1)$ and $\left(4, \frac{1}{2}\right)$.

   (c) Graph the curve and both tangents on a common screen

   Solution

   (a) We need to first take the derivative:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{x - (x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}\right) \\ &= \lim_{h\to 0} \frac{1}{\cancel{h}}\left(\frac{-\cancel{h}}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}\right) \\ &= -\frac{1}{\sqrt{x}\sqrt{x}(2\sqrt{x})} \\ &= -\frac{1}{2}x^{-\frac{3}{2}} \end{aligned}$$

   So plug in $x = a$ to get the slope:

$$f'(a) = -\frac{1}{2}a^{-\frac{3}{2}}$$

   We see that if $a = 0$, this is not valid.

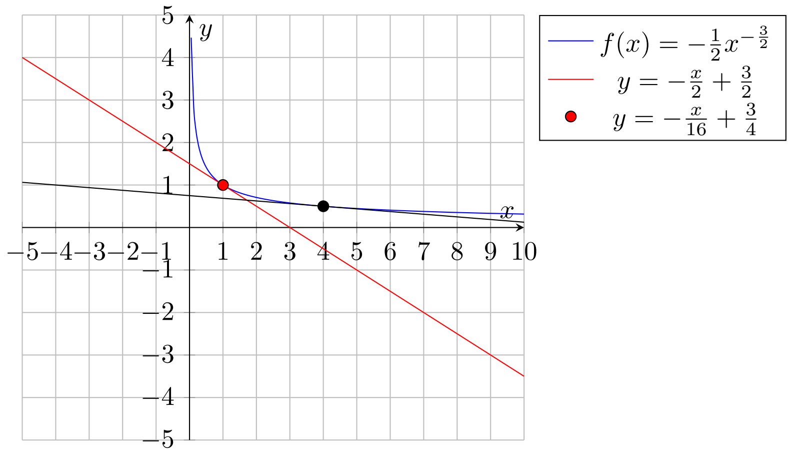
   To get the tangent line at $(1, 1)$, just use $y = mx + b$, where $m = f'(1)$ and we can compute $b = 1 - m$.

$$\begin{aligned} m &= -\frac{1}{2} \\ b &= 1 - \left(-\frac{1}{2}\right) = \frac{3}{2} \\ y &= -\frac{x}{2} + \frac{3}{2} \end{aligned}$$

(b) To get the tangent line at $\left(4, \frac{1}{2}\right)$, do the same thing, but now $m = f'(4)$ and $b = \frac{1}{2} - 4m$.

$$m = -\frac{1}{2}(4)^{-\frac{3}{2}} = -\frac{1}{16}$$
$$b = \frac{1}{2} - \left(-\frac{1}{4}\right) = \frac{3}{4}$$
$$y = -\frac{x}{16} + \frac{3}{4}$$

(c) The graph is below:



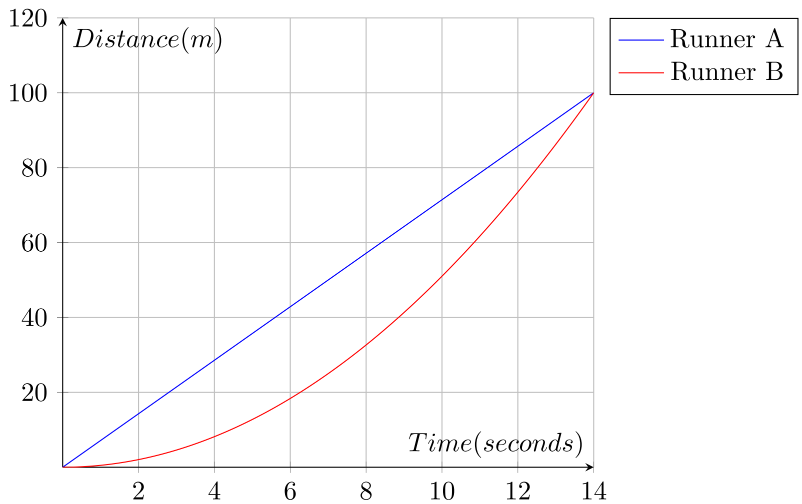
2. If the tangent line to the curve $y = f(x)$ at $(4, 3)$ passes through $(0, 2)$, find $f(4)$ and $f'(4)$.

Solution

We know that $f(4) = 3$ as the tangent curve hits $f(x)$ at $(4, 3)$. Since we have two points through which the tangent line passes, we have a way of finding the slope of that line (that is, $f'(4)$):

$$f'(4) = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3 - 2}{4 - 0} = \frac{1}{4}$$

3. Shown are graphs of the position functions of two runners, A and B, who run a 100-m race and tie:



(a) Describe how each of the runners ran the race.

(b) At what time is the distance between the runners the greatest?

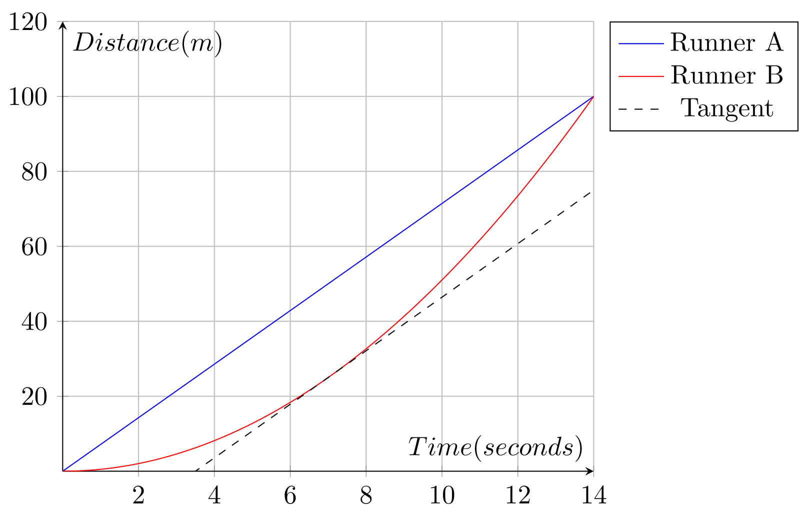
(c) At what time do they have the same velocity?

Solution

4. Runner A ran an even pace, while Runner B ran "negative splits" (sped up as the race went on).

5. The time that the distance between the runners is the greatest is equal to the time that the runners have the same velocity (this will become clear later on in the course as we become more comfortable with the derivative and what it means). We can guess from the graph that the distance is greatest at 7 seconds into the race.

6. The tangent line to runner B's curve at $t = 7$ is shown below, confirming our answer in part (b). Since runner B does not have constant velocity, we need to find the slope of the tangent line to the position curve to find the instantaneous velocity. The tangent line at $t = 7$ is parallel to runner A's trajectory, meaning the velocities are equal.



7. If a ball is thrown into the air with a velocity of 40 ft/s, its height after t seconds is $y = 40t - 16t^2$. Find the velocity when $t = 2$.

Solution: The velocity is simply the derivative of the height function, with increasing height taken as the positive velocity direction (so a falling ball will have negative velocity). Take the derivative:

$$\begin{aligned}\frac{dy}{dt} = v(t) &= \lim_{t\to 0} \frac{y(t+h) - y(t)}{h} \\ &= \lim_{t\to 0} \frac{40(t+h) - 16(t+h)^2 - 40t + 16t^2}{h} \\ &= \lim_{t\to 0} \frac{40h - 32th}{h} \\ &= \lim_{t\to 0} 40 - 32t \\ &= 40 - 32t\end{aligned}$$

Then we substitute $t = 2$ to find the velocity at that time:

$$\begin{aligned}v(2) &= 40 - 32(2) \\ &= 40 - 64 \\ &= -24 \text{ ft/s}\end{aligned}$$

8. Find $f'(a)$:

$$f(t) = \frac{2t+1}{t+3}$$

Solution: Use the definition of the derivative:

$$\begin{aligned}f'(t) &= \lim_{h\to 0} \frac{\frac{2(t+h)+1}{t+h+3} - \frac{2t+1}{t+3}}{h} \\ &= \lim_{h\to 0} \frac{\frac{2t+2h+1}{t+h+3} - \frac{2t+1}{t+3}}{h} \\ &= \lim_{h\to 0} \frac{\frac{(2t+2h+1)(t+3)}{(t+3)(t+h+3)} - \frac{(2t+1)(t+h+3)}{(t+3)(t+h+3)}}{h} \\ &= \lim_{h\to 0} \frac{(2t+1+2h)(t+3) - (2t+1)(t+3+h)}{h(t+3)(t+h+3)} \\ &= \lim_{h\to 0} \frac{2h(t+3) - (2t+1)h}{h(t+3)(t+h+3)} \\ &= \lim_{h\to 0} \frac{2(t+3) - (2t+1)}{(t+3)(t+h+3)} \\ &= \frac{2(t+3) - (2t+1)}{(t+3)^2} \\ &= \frac{5}{(t+3)^2}\end{aligned}$$

9. These limits are the derivative of a function f evaluated at a. Find f and a for each:

   (a) $\lim_{h\to 0} \frac{(1+h)^{10}-1}{h}$

   (b) $\lim_{x\to 5} \frac{2^x-32}{x-5}$

   (c) $\lim_{x\to \frac{\pi}{4}} \frac{\tan(x)-1}{x-\frac{\pi}{4}}$

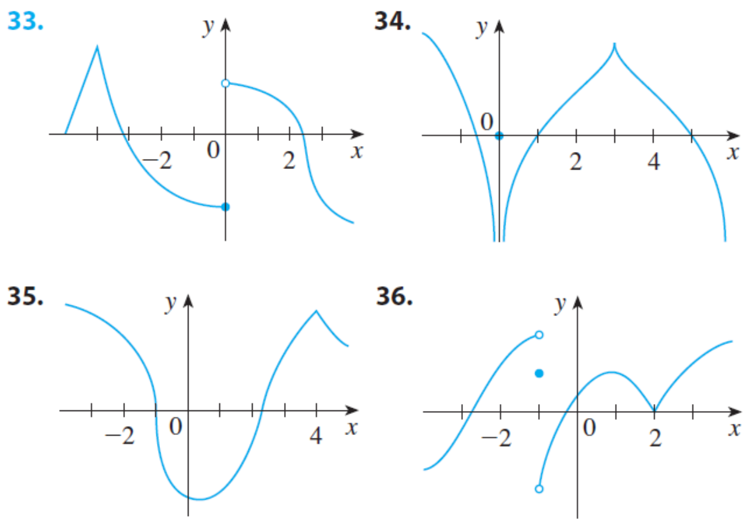
Solution:

(a) $f(x) = x^{10}$, $a = 1$

(b) $f(x) = 2^x$, $a = 5$

(c) $f(x) = \tan(x)$, $a = \frac{\pi}{4}$

10. Picture from Stewart (page 94)



For each graph (33-36, which we'll call (a)-(d)), say where and why f is not differentiable

(a) At $x = -4$ because there is a corner, and at $x = 0$ due to a discontinuity.

(b) At $x = 0$ due to a discontinuity, at $x = 3$ due to a cusp, and at $x = 6$ due to a vertical line (which looks like it will be an asymptote)

(c) At $x = -1$ since it is a fully vertical segment, and at $x = 4$ since there is a corner/cusp

(d) At $x = -1$ due to a discontinuity, and $x = 2$ due to a cusp.

11. Prove each of the following:

(a) The derivative of an even function is an odd function

(b) The derivative of an odd function is an even function

Solution:

(a) Assume $f(x)$ is even. We can use the definition of the derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$f'(-a) = \lim_{x \to -a} \frac{f(-x) - f(-a)}{-x - a}$$

Let $b = -a$:

$$\begin{aligned} f'(b) &= \lim_{x \to b} \frac{f(x) - f(b)}{-x + b} \\ &= \lim_{x \to b} - \left(\frac{f(x) - f(b)}{x - b} \right) \\ &= -f'(b) \end{aligned}$$

So f' is odd.

(b) Assume $f(x)$ is odd. We can use the definition of the derivative:

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ f'(-a) &= \lim_{x \to -a} \frac{f(x) - f(-a)}{x + a} \end{aligned}$$

Let $b = -a$:

$$\begin{aligned} f'(b) &= \lim_{x \to b} \frac{f(x) - f(b)}{x - b} \\ &= \lim_{x \to b} \frac{f(x) - f(b)}{x - b} \\ &= f'(b) \end{aligned}$$