Algebra and Calculus: Worksheet 7 Solutions

Graphing rational functions using transformations:

In general, we can represent rational functions that are ratios of first degree polynomials as transformations of the function $f(x) = \frac{1}{x}$:

$$\begin{aligned}\frac{ax+b}{cx+d} &= \frac{a}{c} + \frac{b - \frac{ad}{c}}{c\left(x + \frac{d}{c}\right)} \\ &= \frac{a}{c} + \left(b - \frac{ad}{c}\right) f\left(c\left[x + \frac{d}{c}\right]\right)\end{aligned}$$

assuming $c \neq 0$. This gives the horizontal and vertical asymptotes immediately:

- horizontal: $y = \frac{a}{c}$
- vertical: $x = -\frac{d}{c}$

The formula is simply the result of dividing $cx + d$ into $ax + b$, and you can always use that strategy if you forget the formula.

1. $t(x) = \frac{3x-3}{x+2}$.

   Solution:

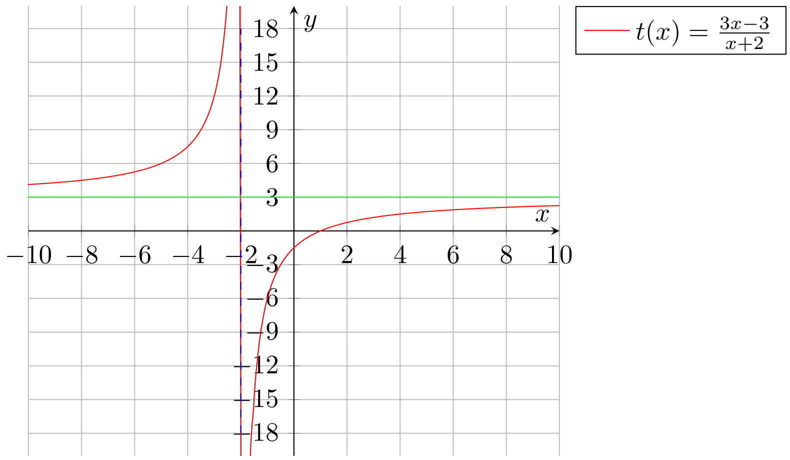
   We can rewrite using the formula above. We have that $a = 3, b = -3, c = 1, d = 2$:

$$\begin{aligned}t(x) &= \frac{3}{1} + \frac{-3 - \frac{3(2)}{1}}{1\left(x + \frac{2}{1}\right)} \\ &= 3 - \frac{9}{x+2} \\ &= 3 - 9f(x+2)\end{aligned}$$

   So we

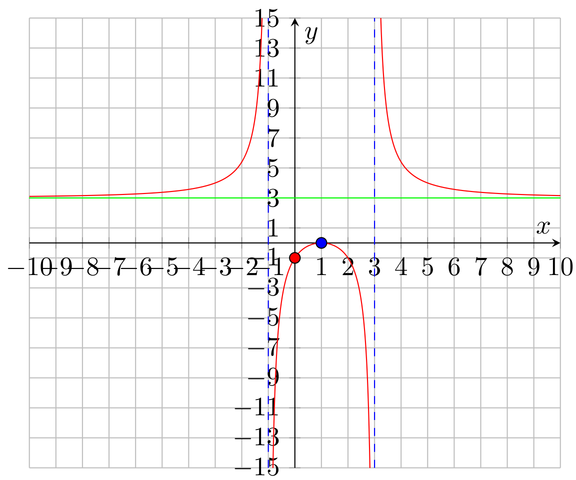
   - Shift $\frac{1}{x}$ two units to the left
   - Flip it about the x-axis
   - Stretch it vertical by nine units
   - Shift it up by three units

   Our horizontal asymptote is at $y = \frac{a}{c} = 3$ and the vertical asymptote is at $x = -\frac{d}{c} = -2$. See the graph:



2. Determine the x and y intercepts and the vertical and horizontal asymptotes.

   The graph is of a rational function equal to the ratio of two quadratic polynomials. Now use all the given information to find the function.



Solution: The x-intercept is simply the dot at $(1, 0)$. The multiplicity is even, and since the rational function is the ratio of two functions, we conclude that the numerator must be proportional to $(x-1)^2$.

The y-intercept is the dot at $(0, -1)$. We need our function, let's call it $r(x)$, to have $r(0) = -1$.

As for asymptotes, we see that there are two vertical asymptotes at $x = 3$ and $x = -1$, and the horizontal asymptote is at $y = 3$. Due to the two vertical asymptotes, we conclude the denominator must be proportional to $(x-3)(x+1)$. So

$$r(x) = c\frac{(x-1)^2}{(x-3)(x+1)}$$

Due to the horizontal asymptote at $y = 3$, we need $\lim_{x\to\infty} r(x) = 3$, which means we must set $c = 3$. Thus

$$r(x) = \frac{3(x-1)^2}{(x-3)(x+1)}$$

and we confirm $r(0) = \frac{3(-1)^2}{(-3)(1)} = \frac{3}{-3} = -1$, which is what the y-intercept should be.

3. Find the slant asymptotes and vertical asymptotes of $r(x) = \frac{x^2}{x-2}$.

   Solution: Let's divide synthetically:

2	1	0	0
	$\downarrow$	2	4
	1	2	4

   So

   $$\frac{x^2}{x-2} = x + 2 + \frac{4}{x-2}$$

   So the slant asymptotes are $y = x + 2$ and there is a vertical asymptote at $x = 2$.

4. Find all relevant quantities (intercepts and asymptotes) and plot the graph:

   $$r(x) = \frac{x^2 + 4x - 5}{x^2 + x - 2}$$

   Solution: We'll proceed as usual. Start by factoring:

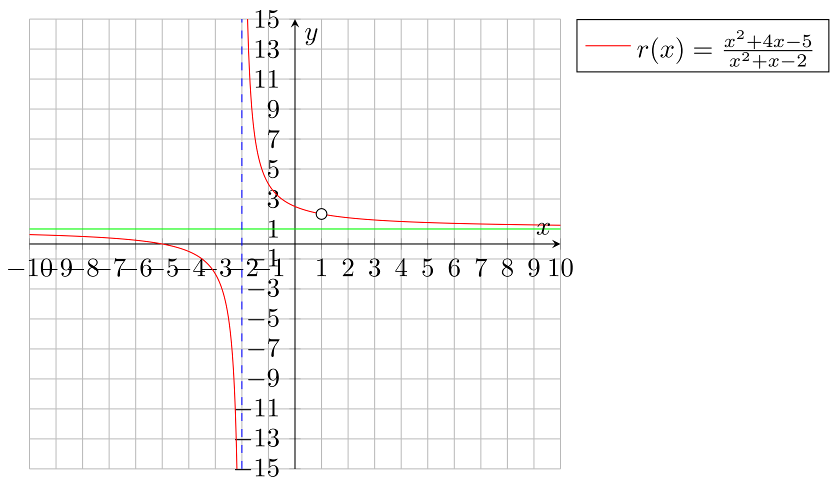
   $$r(x) = \frac{(x+5)(x-1)}{(x+2)(x-1)}$$

   Notice that the numerator and denominator share a factor of $x - 1$. This will result in a hole in the graph at $x = 1$, but there will not be a vertical asymptote there.

   There *is*, however, a vertical asymptote at $x = -2$.

   As for horizontal asymptotes, we look at the end behavior of the polynomials in the numerator and denominator. Because they're both quadratics, we simply take the ratio of the coefficients on the x^2 terms, which is just 1. So there is a horizontal asymptote at $y = 1$.

   The graph looks like:

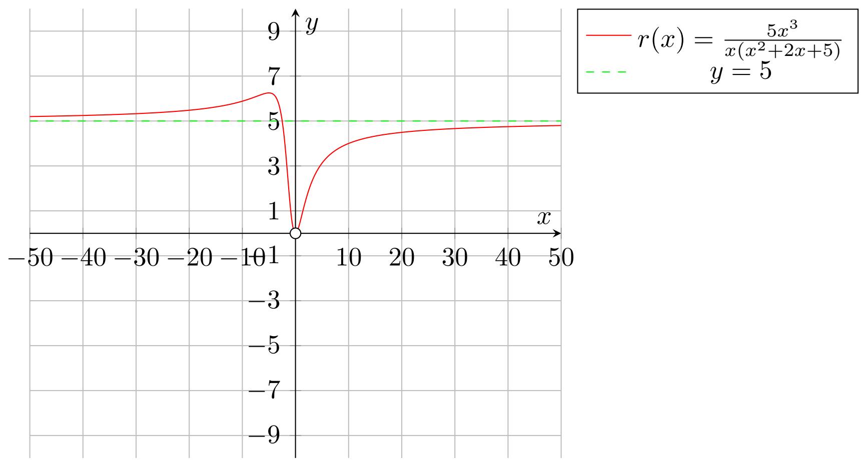
   

5. **True or false**: If there is a vertical asymptote at $x = c$, then the domain must not include the point $x = c$.

   Solution: True. Vertical asymptotes make the denominator of a function equal 0, which is an illegal operation.

6. **True or false**: If there is a horizontal asymptote at $y = d$, then the range must not include the point $y = d$.

   Solution: False. Consider the plot of $r(x) = \frac{5x^3}{x(x^2+2x+5)}$:



   There is a horizontal asymptote at $y = 5$ but in fact $y = 5$ is achieved by the function between the asymptotes as the graph shows.

7. *Q87, Section 3.6*: Suppose that the rabbit population on Mr. Jenkins' farm follows the formula

$$p(t) = \frac{3000t}{t+1}$$

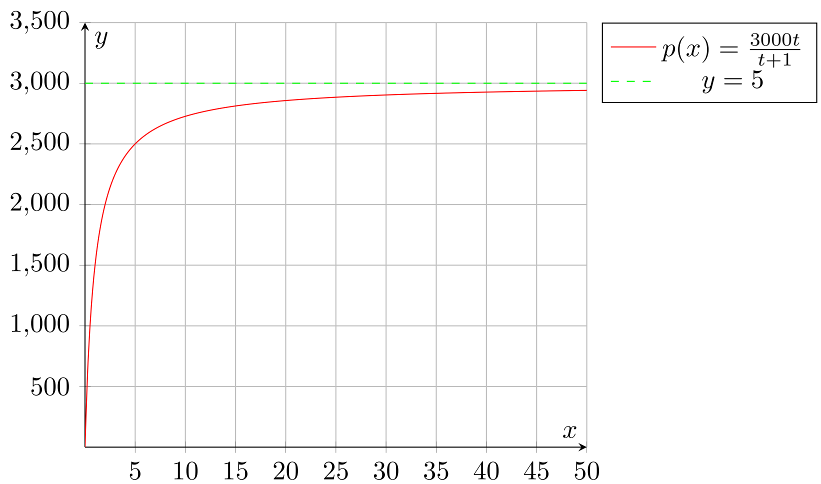
   where $t \geq 0$ is in months.

   (a) Draw a graph of the rabbit population

   (b) What eventually happens to the population?

   Solution:

   (a) The graph looks like:



   The graph answers the question to part (b)

(b) As time goes on, we asymptote to a constant value of 3000 as $t \to \infty$.

8. (*Q94 Section 3.6*): Explain how you can tell (without graphing it) that the function

$$r(x) = \frac{x^6 + 10}{x^4 + 8x^2 + 15}$$

has no x-intercept and no horizontal, vertical, or slant asymptote?

Solution: Note that the powers of x are all even, and so no matter what values of x you put into the function, the numerator and denominator will both be positive (the smallest possible value in the numerator is 10, and the smallest possible value in the denominator is 15).

So the graph never crosses the x-axis. This means there can be no x-intercept.

In order for there to be a horizontal asymptote, the degree of the numerator must be the same degree or less than the degree of the denominator. But the numerator is of higher degree. Thus, there are no horizontal asymptotes.

There cannot be any vertical asymptotes because the denominator never equals 0.

This cannot have any slant asymptotes because the degree of the numerator is more than one degree higher than that of the denominator.